ON THE FOURIER-STIELTJES COEFFICIENTS OF CANTOR-TYPE DISTRIBUTIONS

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ABSTRACT

For a given n-tuple of non-negative numbers $(p_0, p_1, \ldots, p_{n-1})$ whose sum is equal to unity let $\mu(t)$ denote the probability that $\Sigma_{j=1}^{\infty} X_j/n^j \leqq t$, where the independent random variables X_j assume the values $0, 1, \cdots, n-1$ with probabilities $p_0, p_1, \ldots, p_{n-1}$ respectively. For most n-tuples we obtain upper and lower bounds on $|\hat{u}(m)|$; these estimates involve the n-ary representation of m, or in some cases of $2m$, so that a very simple and explicit characterization of the sequences on which $\hat{u}(m)$ approaches zero can be given. In particular, for the Cantor middle-third measure, corresponding to the triple $(1/2, 0, 1/2)$, the following criterion is obtained. $\hat{u}(m)$ approaches zero on a sequence T of integers if and only if $\psi(2m)$ approaches infinity on T, where $\psi(k)$ is the sum of the following three quantities associated with the ternary representation of k: the number of runs of zeros, the number of runs of twos and the number of ones. The results obtained are easily extended to the case when the n-tuple varies with j (subject to certain mild restrictions).

1.

Let μ be any continuous measure on the interval $[0, 1]$. According to a simple but remarkable theorem of Wiener [1, p. 42], the Fourier-Stieltjes (FS) coefficients $(\hat{\mu}j) = \int_0^1 \exp(-2\pi ijt)\, d\mu(t)$ satisfy the limiting relation

$$\frac{1}{n} \sum_{j=1}^{n} |\hat{\mu}(j)|^2 \to 0;$$

in fact, more generally,

$$\frac{1}{n} \sum_{j=1}^{n} |\hat{\mu}(j+k)|^2 \to 0, \text{ uniformly in } k.$$

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It follows readily that for any positive number ε there exists a subset S_ε of the integers† possessing density zero such that $|\hat{\mu}(m)| < \varepsilon$ on the complement of S_ε; *a fortiori*, $\liminf_{m\to\infty} |\hat{\mu}(m)| = 0$, so that one can select a subsequence on which $\hat{\mu}(m)$ approaches zero. (This fact may be looked upon as a generalization of the Riemann-Lebesgue lemma, which holds under the additional hypothesis of *absolute* continuity.) Very little is known concerning the problem of associating with the given measure μ a sequence on which $\hat{\mu}$ approaches zero. (Refer to [4, p. 80], [5, p. 148].) In this note we obtain a solution to this problem for an important class of measures which includes, in particular, the middle-third measure of Cantor.

Let $(p_0, p_1, \cdots, p_{n-1})$ be an n-tuple $(n \geqq 2)$ of non-negative real numbers whose sum equals one, and let μ be the cumulative distribution function of the measure defined on the interval $[0, 1]$ as follows: $\mu(t)$ is the probability that

$$\sum_{j=1}^{\infty} \frac{X_j}{n^j} \leqq t,$$

where the independent random variables $\{X_j\}$ assume the values $0, 1, 2, \cdots, n-1$ with probabilities $p_0, p_1, \cdots, p_{n-1}$ respectively. The n-tuple $(1/n,\ 1/n, \cdots, 1/n)$ corresponds, of course, to Lebesgue measure, while the n-tuples containing $n-1$ zeros correspond to unit mass at a certain point. Aside from these trivial cases, the measure is continuous but singular; refer to [2]. In particular, the triple $(\frac{1}{2}, 0, \frac{1}{2})$ furnishes the Cantor middle-third measure, which is the canonical example of a continuous singular measure whose FS coefficients fail to approach zero; see [3, p. 127]. Accordingly, we may refer to the measures under consideration as being of Cantor type.

In this paper we shall obtain very precise descriptions of the sequences on which $\hat{\mu}(m) \to 0$ for the aforementioned measures and for certain somewhat more general measures.

2.

In the case $n = 2$, the FS coefficients of the measure associated with the pair (p_0, p_1) $(= (1 - p_1, p_1))$ are given by

$$\hat{\mu}(m) = \sum_{j=1}^{\infty} \left(1 - p_1 + p_1 \exp \frac{2\pi i m}{2^j}\right), \tag{1}$$

† Since $\hat{u}(-m) = \overline{\hat{u}(m)}$, we confine attention to the positive integers.

so that

$$(2)\qquad |\hat{\mu}(m)|^2 = \prod_{j=1}^{\infty}\left(1-(1-\alpha)\sin^2\frac{\pi m}{2^j}\right),\qquad \alpha=(2p_1-1)^2.$$

Setting aside the case $\alpha = 1$, corresponding to $p_1 = 0$ or 1, and the case $\alpha = 0$, corresponding to Lebesgue measure, we proceed to obtain upper and lower bounds on $|\hat{\mu}(m)|^2$. From the expansion

$$(3)\qquad -\log\left(1-\left(1-\alpha\right)\sin^2\frac{\pi m}{2^j}\right)=\sum_{k=1}^{\infty}\frac{1}{k}(1-\alpha)^k\sin^{2k}\frac{\pi m}{2^j}$$

we immediately obtain

$$(4)\qquad \begin{aligned}(1-\alpha)\sin^2\frac{\pi m}{2^j} &\leqq -\log\left(1-(1-\alpha)\sin^2\frac{\pi m}{2^j}\right)\\ &\leqq \sum_{k=1}^{\infty}\frac{1}{k}(1-\alpha)^k\sin^2\frac{\pi m}{2^j}\\ &= C\sin^2\frac{\pi m}{2^j},\ C=\log\frac{1}{\alpha}.\end{aligned}$$

Summing over j and taking account of (2), we obtain

$$(5)\qquad (1-\alpha)\sum_{j=1}^{\infty}\sin^2\frac{\pi m}{2^j}\leqq -\log|\hat{\mu}(m)|^2\leqq C\sum_{j=1}^{\infty}\sin^2\frac{\pi m}{2^j}.$$

Thus, $\hat{\mu}(m)$ approaches zero on a sequence T of positive integers if and only if the sum of the series $\sum_{j=1}^{\infty}\sin^2\pi m/2^j$ becomes infinite as m goes to infinity on T. We shall show that the sum of this series is of the order of magnitude $R(m)$, the number of runs (maximal blocks of the same digit) appearing in the binary representation of m; that is, there exist two positive constants C_1, C_2 such that, for all (positive) integers m,

$$(6)\qquad C_1R(m) < \sum_{j=1}^{\infty}\sin^2\frac{\pi m}{2^j} < C_2R(m).$$

Rather than giving a formal proof of this fact, which would be tedious, an illustrative example will clarify the whole idea. Let m be the number possessing the binary representation 1100111 (thus, $m = 103$). Corresponding to the right-most run, we obtain the quantities $\sin^2\frac{1}{2}\pi$, $\sin^2\frac{1}{4}\pi$, $\sin^2\frac{1}{8}\pi$, which are dominated respectively by $\pi^2/2^2$, $\pi^2/4^2$, $\pi^2/8^2$, and exceed, respectively, $(2/\pi)^2\cdot\pi^2/2^2$, $(2/\pi)^2\cdot\pi^2/4^2$, $(2/\pi)^2\cdot\pi^2/8^2$. Thus, the contribution of this run to the sum $\sum_{j=1}^{\infty}\sin^2\pi m/2^j$ is less than $\pi^2(1/2^2+1/4^2+1/8^2+\cdots)$ and greater than unity. A trivial modification holds for the other two runs; the lower and upper bounds obtained are weaker than those obtained above, but this is of no concern. By addition one obtains (6) with $R(m) = 3$. This argument holds equally well for any *odd* number m, while for *even* values of m we note that, if m' is the largest odd divisor of m, the sum $\sum_{j=1}^{\infty} \sin^2 \pi m / 2^j$ is unchanged when m is replaced by m', while $R(m) = 1 + R(m')$. Therefore, the proof of (6) may be considered complete.

Thus we have proved the following result.

THEOREM 1. *Let $p_1 \neq 0, \frac{1}{2}, 1$ and let μ be the measure determined by the pair (p_0, p_1). Then $\hat{\mu}(m)$ approaches zero on the sequence T if and only if, on this sequence, the number of runs in the binary representation of m becomes infinite.*

We shall now exhibit a set S of density zero on whose complement $R(m) \to \infty$; it will then follow from (6) that $\hat{\mu}(m)$ approaches zero on the complement of S for *every* measure μ of the type under consideration. Let S consist of those integers for which $R(m) < \frac{1}{4} K(m)$, where $K(m)$ is the length of the binary representation of m, that is, $K(m) = 1 + [\log_2 m]$. If $S(m)$ denotes the number of integers not exceeding m which belong to S, then clearly

$$\frac{S(m)}{m} \leqq \frac{S(2^{K(m)})}{2^{K(m)-1}}. \tag{7}$$

Now a completely elementary argument shows that $S(2^n) = o(2^n)$, and hence $S(m) = o(m)$. Thus, we have proved Theorem 2.

THEOREM 2. *Let S be the set of positive integers whose binary representation satisfies the condition $R(m) < \frac{1}{4} K(m)$, where $R(m)$ denotes the number of runs and $K(m)$ denotes the number of digits, and let μ be the measure determined by the pair $(1 - p_1, p_1)$, with $p_i \neq 0, \frac{1}{2}, 1$. Then S is of density zero, and $\hat{\mu}(m)$ approaches zero on the complement of S. (Clearly, the factor $\frac{1}{4}$ appearing in the inequality may be replaced by any positive number smaller than $\frac{1}{2}$.)*

The set S which has been defined above is not of *uniform* density zero, for the fact that $R(m+1) = R(m) \pm 1$ immediately shows that S contains increasingly long successions of numbers which include the powers of 2 (for which numbers the binary representation contains exactly two runs). From Theorem 1 it is evident that, given *any* positive integer n and any set of integers $\tilde{S}$ on whose complement $\hat{\mu}(m) \to 0$, $\tilde{S}$ must contain, infinitely often, successions of length greater than n. Thus, it is impossible to thin out the set S so as to obtain a set of *uniform* density zero.

By a minor extension of the foregoing arguments we can extend the results to a broader class of continuous singular measures, as follows.

THEOREM 3. *Let $a_1, a_2, a_3, \cdots$ be a sequence of numbers satisfying* lim sup $|a_j - \frac{1}{2}| < \frac{1}{2}$, *and let $\mu(t)$ = probability that $\sum_{j=1}^{\infty} X_j/2^j \leqq t$, where the independent random variables $\{X_j\}$ assume the values 0, 1 with probabilities a_j, $1 - a_j$ respectively. Then $\hat{\mu}(m) \to 0$ on any set of integers on which $R(m) \to \infty$, in particular on the complement of the set S defined above; if, in addition,* $\liminf |a_j - \frac{1}{2}| > 0$, *then $\hat{\mu}(m) \to 0$ on a set T if* and only if *$R(m) \to \infty$ on T.*

3.

In this and the following sections we extend the previous results to the measures determined by triples (p_0, p_1, p_2). In analogy with (1) and (2) we have

$$\hat{\mu}(m) = \prod_{j=1}^{\infty} \left(p_0 + p_1 \exp\left(-\frac{2\pi i m}{3^j}\right) + p_2 \exp\left(-\frac{4\pi i m}{3^j}\right)\right) \tag{7}$$

and

$$|\hat{\mu}(m)|^2 = \prod_{j=1}^{\infty} \left(1 - 4p_1(p_0 + p_2)\sin^2\frac{\pi m}{3^j} - 4p_0p_2\sin^2\frac{2\pi m}{3^j}\right). \tag{8}$$

First we consider the triple $(\frac{1}{2}, 0, \frac{1}{2})$, corresponding to Cantor's middle-third measure. In this case (8) reduces to

$$|\hat{\mu}(m)|^2 = \prod_{j=1}^{\infty} \left(1 - \sin^2\frac{2\pi m}{3^j}\right), \tag{9}$$

from which it is apparent that $\hat{\mu}(m) \neq 0$, $\hat{\mu}(3m) = \hat{\mu}(m)$; this suffices to show that $\hat{\mu}(m)$ fails to converge to zero, and that in estimating its magnitude we may confine attention to values of m which are not divisible by 3, that is, whose ternary representation terminates in 1 or 2.

Now, let $2m$ (*not* m) possess the ternary representation $a_1 a_2 \cdots a_K$, where $K = 1 + [\log_3 2m]$. Imagine this representation broken up into runs of zeros, ones, and twos; we can then write the above representation in the symbolic form

$$2m = \alpha_1^{l_1}\alpha_2^{l_2} \cdots \alpha_R^{l_R}, \tag{10}$$

where R denotes the total number of runs, $l_1, l_2, \cdots, l_R$ denote the lengths of these runs, and each α is one of the digits 0,1,2, no two adjacent alphas being equal and

α_1 and α_R being 1 or 2. In (9) we break the product into $R+1$ subproducts; in these subproducts j ranges, respectively, from 1 to l_R, from $1+l_R$ to $l_{R-1}+l_R$, from $1+l_{R-1}+l_R$ to $l_{R-2}+l_{R-1}+l_R$, etc. (In the final subproduct, of course, j ranges from $K+1$ to infinity.)

We begin by obtaining upper and lower bounds on the first of these subproducts,

$$P_1 = \prod_{j=1}^{l_R} \left(1 - \sin^2\frac{2\pi m}{3^j}\right) = \prod_{j=1}^{l_R} \cos^2 \frac{2\pi m}{3^j}. \tag{11}$$

If $\alpha_R = 2$, then P_1 is immediately seen to assume the form

$$P_1 = \prod_{j=1}^{l_R} \cos^2 \frac{\pi}{3^j}, \tag{12}$$

and so

$$0 < \prod_{j=1}^{\infty} \cos^2\frac{\pi}{3^j} < P_1 < 1, \tag{13}$$

or

$$0 < -\log P_1 < -\log\left(\prod_{j=1}^{\infty} \cos^2 \frac{\pi}{3^j}\right). \tag{14}$$

On the other hand, if $\alpha_R = 1$, then instead of (12) we obtain

$$P_1 = \prod_{j=1}^{l_R} \sin^2\frac{\pi}{2\cdot 3^j}, \tag{15}$$

and so (using the inequalities $1/3^j < \sin \pi/2\cdot 3^j < \pi/2\cdot 3^j$) we obtain

$$-2l_R\log\tfrac{1}{2}\pi + 2l_R(l_R+1)\log 3 < -\log P_1 < 2l_R(l_R+1)\log 3. \tag{16}$$

Now, for each of the other *finite* products, the above estimates carry over with minor modifications whenever a run of ones or twos is under consideration, while a run of zeros furnishes exactly the same pair of estimates as a run of twos. Finally, the infinite subproduct (corresponding to $j > K = l_1 + l_2 + \cdots + l_R$) corresponds to a run of zeros.

Summing up, we conclude that for each run of zeros and each run of twos the corresponding subproduct P_k (including the infinite subproduct) satisfies

$$C_1 < -\log P_k < C_2, \tag{17}$$

while for each run of ones we have, for the corresponding subproduct,

$$C_1(\text{length of run})^2 < -\log P_k < C_2\,(\text{length of run})^2, \tag{18}$$

where C_1 and C_2 are suitably chosen *universal* constants. Summing over all the runs, we obtain

$$C_1(R_0m) + R_2(2m) + S_1(2m)) < \log |\hat{\mu}(m)|^2 < C_2(R_0(2m) + R_2(2m) + S_1(2m)), \tag{19}$$

where

$R_0(k)$ = number of 0-runs in ternary representation of k;

$R_2(k)$ = number of 2-runs in ternary representation of k;

$S_1(k)$ = sum of squares of length of 1-runs in ternary representation of k.

If we denote the number of appearances of the digit 1 in the ternary representation of k by $N_1(k)$, then obviously $N_1(k) \leqq S_1(k) \leqq N_1(k)^2$; for convenience both here and later, we introduce the notation

$$\begin{aligned} \phi(k) &= R_0(k) + R_2(k) + S_1(k), \\ \psi(k) &= R_0(k) + R_2(k) + N_1(k). \end{aligned} \tag{20}$$

Then we can sum up our results as follows.

THEOREM 4. *The* FS *coefficients of the Cantor middle-third measure satisfy the inequalities*

$$\exp(-C_2\phi(2m)) < |\hat{\mu}(m)|^2 < \exp(-C_1\phi(2m)) \tag{21}$$

for suitably chosen positive constants C_1, C_2. Thus, $\hat{\mu}(m)$ approaches zero on a sequence T if and only if $\phi(2m)$ or (equivalently) $\psi(2m)$ approaches infinity on T.

As in the preceding section, one can now easily demonstrate the existence of a set S of density zero on whose complement the condition stated in the theorem is satisfied. Let S be the set of integers m which satisfy the condition

$$N_1(2m) < \tfrac{1}{4}(1 + [\log_3 2m]). \tag{22}$$

Then, very much as in the preceding section, one shows that S is of density zero, and Theorem 4 guarantees that $\hat{\mu}(m)$ approaches zero on the complement of S.

4.

Let us momentarily replace, in (7) and (8), $2\pi m/3^j$ by a continuous real variable u. Then it is evident that

$$\max[4p_1(p_0 + p_2)\sin^2 u + 4p_0p_2\sin^2 2u] \leqq 1,$$

and that equality holds if and only if $p_0 + p_1 e^{-iu} + p_2 e^{-2iu}$ can vanish. This condition furnishes the pair of *real* equations

$$p_1 + (p_0 + p_2)\cos u = 0, \qquad (p_0 + p_2)\sin u = 0. \tag{23}$$

This system possesses a solution if and only if either (or both) of the following conditions are satisfied:

$$p_0 = p_2 \geqq \tfrac{1}{4}, \tag{24a}$$

$$p_1 = \tfrac{1}{2}. \tag{24b}$$

Setting aside these conditions for later consideration, we see that, exactly as in Section 2, $\hat{\mu}(m)$ approaches zero on the sequence T if and only if, on T,

$$\sum_{j=1}^{\infty} \left[p_1(p_0 + p_2)\sin^2 \frac{\pi m}{3^j} + p_0 p_2 \sin^2 \frac{2\pi m}{3^j} \right] \to \infty. \tag{25}$$

We now set aside for later consideration, in addition to (24), the additional case

$$p_1 = 0 \text{ or } 1. \tag{26}$$

Then (25) is certainly implied by

$$\sum_{j=1}^{\infty} \sin^2 \frac{\pi m}{3^j} \to \infty; \tag{27}$$

conversely,

$$\begin{aligned} & p_1(p_0 + p_2)\sin^2 \frac{\pi m}{3^j} + p_0 p_2 \sin^2 \frac{2\pi m}{3^j} \\ & = \{p_1(p_0 + p_2) + 4p_0 p_2\}\sin^2 \frac{\pi m}{3^j} - 4p_0 p_2 \sin^4 \frac{\pi m}{3^j} \\ & \leqq \{p_1(p_0 + p_2) + 4p_0 p_2\}\sin^2 \frac{\pi m}{3^j}, \end{aligned} \tag{28}$$

and so (25) implies (27). Thus, subject to the various restrictions that have been imposed, $\hat{\mu}(m)$ approaches zero on T if and only if (27) holds. Now, by a virtual repetition of the argument presented in Section 3, we can show that, for suitable positive constants C_1 and C_2,

$$C_1 \psi(m) < \sum_{j=1}^{\infty} \sin^2 \frac{\pi m}{3^j} < C_2 \psi(m), \tag{29}$$

and so we obtain the following analogue of Theorem 4.

THEOREM 5. *Let none of the conditions* (24*a*), (24*b*), (26) *be satisfied. Then there exist positive constants* C_1, C_2 *(depending on the triple* (p_0, p_1, p_2)*) such that*

$$\exp(-C_2\psi(m)) < |\hat{\mu}(m)|^2 < \exp(-C_1\psi(m)). \tag{30}$$

Thus, $\hat{\mu}(m)$ approaches zero on T if and only if $\psi(m)$ approaches infinity on this sequence.

It should be emphasized that in Theorem 4 the ternary representation of $2m$, rather than of m, is involved, and that the magnitude of $\hat{\mu}(m)$ is controlled by $\phi(2m)$, while in Theorem 5 the ternary representation of m itself is involved, and $\psi(m)$, not $\phi(m)$, appears in the estimate of $|\hat{\mu}(m)|$.

5.

We proceed to consider the cases there were set aside in Section 4.

First of all, the triples (1,0,0), (0,1,0), (0,0,1) correspond to unit masses at $0, \frac{1}{2}, 1$ respectively, so that $|\hat{\mu}(m)| \equiv 1$.

Secondly, for the triples $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(0, \frac{1}{2}, \frac{1}{2})$ we immediately obtain

$$|\hat{\mu}(m)|^2 = \prod_{j=1}^{\infty} \left(1 - \sin^2\frac{\pi m}{3^j}\right), \tag{31}$$

and by comparison with (9) we see that for the measures corresponding to these two triples the behavior of $|\hat{\mu}(m)|$ is described by Theorem 4 (*not* Theorem 5), except that $2m$ is replaced by m.

Next, for any triple of the form $(p_0, 0, 1 - p_0)$ with $p_0 \neq 0, \frac{1}{2}, 1$, we obtain

$$|\hat{\mu}(m)|^2 = \prod_{j=1}^{\infty} \left(1 - \alpha \sin^2 \frac{2\pi m}{3^j}\right), \qquad 0 < \alpha < 1, \tag{32}$$

and so we have the situation analyzed in Section 4, except that m is replaced by $2m$; thus (30) holds with $\psi(m)$ replaced by $\psi(2m)$.

For the triple $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ we have

$$|\hat{\mu}(m)|^2 = \prod_{j=1}^{\infty} \left(1 - \sin^2\frac{\pi m}{3^j}\right)^2; \tag{33}$$

except for the exponent on the right side, we have the same situation that holds for the triples $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(0, \frac{1}{2}, \frac{1}{2})$. On the other hand for the triples $(p_0, \frac{1}{2}, \frac{1}{2}, p_0)$, with $p_0 \neq 0, \frac{1}{4}, \frac{1}{2}$ we obtain

$$|\hat{\mu}(m)|^2 = \prod_{j=1}^{\infty} \left(1 - \sin^2\frac{\pi m}{3^j}\right)\left(1 - \alpha \sin^2 \frac{\pi m}{3^j}\right), \qquad 0 < \alpha < 1, \tag{34}$$

and so we have a combination, so to speak, of the second and third cases described by (31) and (32) respectively. Once again we conclude that $\hat{\mu}(m)$ approaches zero on T if and only if $\psi(m)$ becomes infinite.

Finally, there remain the triples $(p_0, 1-2p_0, p_0)$, $\frac{1}{4} < p_0 < \frac{1}{2}$. In this case we obtain

$$(35) \qquad |\hat{\mu}(m)|^2 = \prod_{j=1}^{\infty} \left(1 - \alpha \sin^2 \frac{\pi m^2}{3^j}\right), \qquad 1 < \alpha = 4p_0 < 2.$$

One can easily construct a single sequence of density zero on whose complement $\hat{\mu}(m)$ approaches zero for every measure of this type, However it is clear that a necessary condition cannot be given which is *independent of* p_0 on a set T such that $\hat{\mu}(m)$ should approach zero on T since the triplet $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ corresponds to Lebesgue measure, while for any other triplet of this type we have $\hat{\mu}(3^k) = \hat{\mu}(1) \neq 0$. Perhaps a necessary and sufficient condition on T independent of p_0 can be given for all triplets under consideration excepting $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, but we do not investigate this question here.

6.

Now we sketch very briefly the case $n > 3$. It is almost obvious that the previous arguments can be extended to furnish the following result.

THEOREM 6. *Suppose that all components of the n-tuple* $(p_0, p_1, \cdots, p_{n-1})$ *are non-vanishing and that the polynomial* $p_0 + p_1 z + p_2 z^2 + \cdots + p_{n-1} z^{n-1}$ *does not vanish on* $|z| = 1$. *Then equation* (30) *holds, where* $\psi(m)$ *is now interpreted as the sum of*:

$R_0(m) =$ *number of runs of zeros in n-ary representation of m*;

$R_{n-1}(m) =$ *number of runs of* $(n-1)$*s in n-ary representation of m*;

$N(m) =$ *number of digits other than* 0 *and* $(n-1)$ *in n-ary representation of m.*

Continuing in this vein, we also see that if S is now defined as the sequence of integers m for which

$$(36) \qquad N(m) < \tfrac{1}{4}(1 + [\log_n m]),$$

then S is of density zero and $\hat{\mu}(m)$ *approaches zero on the complement of S*; *as before, S cannot be replaced by a set of* uniform *density zero.*

Theorem 3 also generalizes, as follows.

THEOREM 7. *Let A be the subset of the plane* $p_0 + p_1 + p_2 + \cdots p_{n-1} = 1$

consisting of points which satisfy the hypothesis of the preceding theorem. Consider a sequence $(p_0(j), p_1(j), p_2(j), \cdots, p_{n-1}(j))$, $j = 1, 2, 3, \cdots$, *of n-tuples which lie in a compact subset of A, and let* $\mu(t)$ *be the probability that* $\sum_{j=1}^{\infty} X_j / n^j \leqq t$, *where the independent random variables* $\{X_j\}$ *assume the values* $0, 1, 2, \cdots, n-1$ *with probabilities* $p_0(j), p_1(j), \cdots, p_{n-1}(j)$. *Then the conclusion of Theorem* 6 *still holds (so that* $\hat{\mu}(m)$ *approaches zero on the complement of the set S defined above).*

The set A is obviously open (relative to the plane $p_0 + p_1 + p_2 + \cdots + p_{n-1} = 1$) and non-void (for it contains, in particular, any n-tuple with positive components one of which exceeds $\frac{1}{2}$). In the case $n = 3$ we have given an explicit characterization of A, and it would appear to be of interest to extend this characterization to larger values of n.

REFERENCES

1. Y. Katznelson, *Introduction to Harmonic Analysis*, John Wiley and Sons, 1968.
2. G. Marsaglia, *Random variables with independent binary digits*, Ann. Math. Statist **42** (1971), 1922–1929.
3. R. E. Edwards, *Fourier Series* (Vol. 2), Holt, Rinehart and Winston, 1967.
4. R. Salem, *On singular monotonic functions of the Cantor types*, J. Math. and Phys. **21** (1942), 69–82.
5. A. Zygmund, *Trigonometrical Series*, 2nd ed. (Vol. 2), Cambridge University Press, 1959.

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